Ideal Fréchet–Urysohn property of a space of continuous functions

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Applications

(1) Gerlits–Nagy problem on γ and δ.

(2) Borodulin-Nadzieja–Farkas pseudointersection number $\mathfrak{p}_{\mathrm{K}}(\mathcal{J})$.

A topological space Z.

Subset $A \subseteq Z$.

Point $a \in \overline{A}$.

Is there a sequence $\langle a_n : n \in \omega \rangle$ in A converging to a?

Z is a metric space: YES

- take a_n from a ball centered in a and radius $\frac{1}{2^n}$

a has a countable base of neighbourhoods (Z first countable): YES

A topological space Z.

Subset $A \subseteq Z$.

Point $a \in \overline{A}$.

Is there a sequence $\langle a_n : n \in \omega \rangle$ in A converging to a?

A topological space Z has **Fréchet-Urysohn property** if for any subset $A \subseteq Z$ and point $a \in \overline{A}$ there is a sequence $\langle a_n : n \in \omega \rangle$ in A converging to a.

Metric space and first countable space have Fréchet-Urysohn property.

Topology on a family of functions

${}^X\mathbb{R}$ denotes the family of all real-valued functions on X.

${}^X\mathbb{R}$ may be equipped with topology such that a sequence of functions $\langle f_n : n \in \omega\rangle$ converges to function f if and only if it converges pointwisely, i.e., $\langle f_n(x) : n \in \omega\rangle$ converges to $f(x)$ in each $x \in X$.

`topology of pointwise convergence = Tychonoff product topology`

X a topological space.

$\mathrm{C}_p(X)$ denotes the family of all continuous functions on X.

Inherited topology on $\mathrm{C}_p(X) \subseteq {}^X\mathbb{R}$.

If X is a discrete topological space then $\mathrm{C}_p(X) = {}^X\mathbb{R}$.

Basic open neighbourhood of $\mathbf{0}$ in $\mathrm{C}_p(X)$. $\varepsilon > 0$, $x_0, \ldots, x_k \in X$

$$[\varepsilon; x_0, \ldots, x_k] = \{g \in \mathrm{C}_p(X) : \; |g(x_0)| < \varepsilon, \ldots, |g(x_k)| < \varepsilon\}$$

We assume $X \subseteq \mathbb{R}$.

Does $\mathrm{C}_p(X)$ possess Fréchet-Urysohn property?

What is the role of X in $\mathrm{C}_p(X)$ possessing Fréchet-Urysohn property?

Local property of $\mathrm{C}_p(X)$.

$\uparrow\downarrow$

Property of X.

Gerlits J. and Nagy Zs., *Some properties of $C_p(X)$, I,* Topology Appl. **14** (1982), 151–161.

Theorem 2. *Let X be a Tychonoff-space. If $E = C(X)$, then the following implications are valid.*

$$
\begin{array}{lccccccccccccc}
E = C(X) & (i) & \Rightarrow & (ii) & \Rightarrow & (iii) & \Leftrightarrow & (iv) & \Rightarrow & (v) & \Rightarrow & (vi) \\
 & \Updownarrow & & \Updownarrow & & \searrow & & \nearrow & & \Downarrow & & \Updownarrow \\
X & (\alpha) & \Rightarrow & (\beta) & \Rightarrow & & (\gamma) & & \Rightarrow & (\delta) & \Rightarrow & (\varepsilon)
\end{array}
$$

What is the role of X in $C_p(X)$ possessing Fréchet-Urysohn property?

$C_p(X)$ has **Fréchet-Urysohn property** if for any subset $A \subseteq C_p(X)$ and function $f \in \overline{A}$ there is a sequence $\langle f_n : n \in \omega \rangle$ in A converging to f.

Basic open neighbourhood of $\mathbf{0}$ in $\mathrm{C}_p(X)$. $\varepsilon > 0$, $x_0, \ldots, x_k \in X$

$$[\varepsilon; x_0, \ldots, x_k] = \{g \in \mathrm{C}_p(X) : |g(x_0)| < \varepsilon, \ldots, |g(x_k)| < \varepsilon\}$$

$\mathrm{C}_p(X)$ has **Fréchet-Urysohn property** if and only if for any subset $A \subseteq \mathrm{C}_p(X)$ and function $\mathbf{0} \in \overline{A}$ there is a sequence $\langle f_n : n \in \omega \rangle$ in A **converging to $\mathbf{0}$**.

$\{f_n : n \in \omega\} \subseteq \mathrm{C}_p(X)$, $\varepsilon > 0$

$$\boxed{V_n = \{x \in X : |f_n(x)| < \varepsilon\}}$$

- $\mathbf{0} \in \overline{\{f_n : n \in \omega\}}$

$\{V_n : n \in \omega\}$ forms a cover of X, called ω-cover, i.e., each finite subset of X is covered by some V_n.

- $\langle f_n : n \in \omega \rangle$ converges pointwisely to $\mathbf{0}$

$\{V_n : n \in \omega\}$ forms a cover of X, called γ-cover, i.e., each $x \in X$ is an element of V_n for all but finitely many n.

X is **a γ-set** if any **open ω-cover of X** contains **an open γ-subcover of X**.

Does $\mathrm{C}_p(X)$ possess Fréchet-Urysohn property?

Theorem (Gerlits–Nagy 1982)

$\mathrm{C}_p(X)$ possesses Fréchet-Urysohn property if and only if X is a γ-set.

Theorem (Galvin–Miller 1984)

- *If $|X| < \mathfrak{p}$ then X is a γ-set.*
- *If $\mathfrak{p} = \mathfrak{c}$ then there is a γ-set of cardinality $\mathfrak{c}$.*

Theorem (Gerlits–Nagy 1982)

X is a γ-set then X has strong measure zero.

If Borel Conjecture holds then X is a γ-set if and only if X is countable.

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Theorem 2. *Let X be a Tychonoff-space. If $E = C(X)$, then the following implications are valid.*

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 & \Updownarrow & & \Updownarrow & & \nwarrow\!\!\searrow & & \nearrow\!\!\swarrow & & \Downarrow & & \Updownarrow \\
X & (\alpha) & \Rightarrow & (\beta) & \Rightarrow & & (\gamma) & & \Rightarrow & (\delta) & \Rightarrow & (\varepsilon)
\end{array}$$

Problem. Is $(\delta) \Rightarrow (\gamma)$ true (in ZFC)? Is there a model of ZFC in which $(\delta) \Rightarrow (\gamma)$ does not hold?

Problem listed as open:

Miller A., *On γ-sets,* plenary lecture, Second Workshop on Coverings, Selections, and Games in Topology, Lecce, Italy, 19–22 Dec 2005. Lecture notes: `http://u.cs.biu.ac.il/~tsaban/SPMC05/Miller.pdf`.

Orenshtein T. and Tsaban B., *Pointwise convergence of partial functions: The Gerlits–Nagy Problem,* Adv. Math. **232** (2013), 311–326.

Our answer: No. Yes, any model of $\mathfrak{p} = \mathfrak{c}$.

$$\boxed{\{V_n:\ n\in\omega\}\subseteq X}$$

$\{V_n:\ n\in\omega\}$ is γ-cover if each $x\in X$ is an element of V_n for all but finitely many n.

$$\boxed{\underline{\mathrm{Lim}}\{V_n:\ n\in\omega\}=\bigcup_{n\in\omega}\bigcap_{m\geq n}V_m=\{x\in X:\ (\forall^\infty n\in\omega)\ x\in V_n\}}$$

$\{V_n:\ n\in\omega\}$ is a γ-cover if and only if $X=\underline{\mathrm{Lim}}\{V_n:\ n\in\omega\}$.

$$\boxed{\underline{\mathrm{Lim}}\{V_n :\ n \in \omega\} = \bigcup_{n\in\omega} \bigcap_{m\geq n} V_m = \{x \in X :\ (\forall^\infty n \in \omega)\ x \in V_n\}}$$

$$\begin{aligned}
\mathrm{L}^0(\{V_n :\ n \in \omega\}) &= \{V_n :\ n \in \omega\} \\
\mathrm{L}^1(\{V_n :\ n \in \omega\}) &= \{\underline{\mathrm{Lim}}\{W_n :\ n \in \omega\} :\ (\forall n \in \omega)\ W_n \in \mathrm{L}^0(\{V_i :\ i \in \omega\})\} \\
\mathrm{L}^2(\{V_n :\ n \in \omega\}) &= \{\underline{\mathrm{Lim}}\{W_n :\ n \in \omega\} :\ (\forall n \in \omega)\ W_n \in \mathrm{L}^1(\{V_i :\ i \in \omega\})\} \\
\mathrm{L}^\xi(\{V_n :\ n \in \omega\}) &= \{\underline{\mathrm{Lim}}\{W_n :\ n \in \omega\} :\ (\forall n \in \omega)\ W_n \in \bigcup_{\eta<\xi} \mathrm{L}^\eta(\{V_i :\ i \in \omega\})\}
\end{aligned}$$

$\mathrm{L}^{\omega_1}(\{V_n :\ n \in \omega\})$ is the smallest family containing $\mathcal{V}$ and closed under operator $\underline{\mathrm{Lim}}$.

X is a γ-set $\Leftrightarrow$ for any open ω-cover $\mathcal{V}$, $X \in \mathrm{L}^1(\mathcal{V})$

X is a δ-set $\Leftrightarrow$ for any open ω-cover $\mathcal{V}$, $X \in \mathrm{L}^{\omega_1}(\mathcal{V})$

Gerlits J. and Nagy Zs., *Some properties of $C_p(X)$, I,* Topology Appl. **14** (1982), 151–161.

Theorem 2. *Let X be a Tychonoff-space. If $E = C(X)$, then the following implications are valid.*

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Problem. Is $(\delta) \Rightarrow (\gamma)$ true (in ZFC)? Is there a model of ZFC in which $(\delta) \Rightarrow (\gamma)$ does not hold?

Theorem (Orenshtein–Tsaban 2013)
X is a δ-set if and only if $\overline{A} \subseteq \mathtt{partlims}(A)$ for each $A \subseteq \mathrm{C}_p(X)$.

Theorem (Galvin–Miller 1984)
- *If $|X| < \mathfrak{p}$ then X is a γ-set.*
- *If $\mathfrak{p} = \mathfrak{c}$ then there is a γ-set of cardinality $\mathfrak{c}$.*

Theorem (Gerlits–Nagy 1982)
X is a δ-set then X has strong measure zero.

If Borel Conjecture holds then X is a δ-set if and only if X is countable.

$f \in \overline{A} \subseteq \mathrm{C}_p(X)$

Is there a way to describe f via a sequence of elements of A?

Theorem (Arkhangel'skii 1976)

$\mathrm{C}_p(X)$ has countable tightness, i.e., for any $f \in \overline{A} \subseteq \mathrm{C}_p(X)$ there is countable $B \subseteq A$ such that $f \in \overline{B}$.

A family $\mathcal{K} \subseteq \mathcal{P}(\omega)$ is called an ideal if

a) $B \in \mathcal{K}$ for any $B \subseteq A \in \mathcal{K}$,

b) $A \cup B \in \mathcal{K}$ for any $A, B \in \mathcal{K}$,

c) $\mathtt{Fin} = [\omega]^{<\omega} \subseteq \mathcal{K}$,

d) $\omega \notin \mathcal{K}$.

$\mathcal{I}$, $\mathcal{J}$, $\mathcal{K}$ are ideals in the following.

$\mathcal{A} \subseteq \mathcal{P}(\omega)$ $\qquad$ $\mathcal{A}^* = \{A \subseteq \omega : \omega \setminus A \in \mathcal{A}\}$

$\mathcal{F} \subseteq \mathcal{P}(\omega)$ is a filter if $\mathcal{F}^*$ is an ideal.

A sequence of functions $\langle f_n : n \in \omega \rangle$ converges pointwisely to function f if

$$\langle f_n(x) : n \in \omega \rangle \text{ converges to } f(x) \text{ in each } x \in X,$$

$$\{n : |f_n(x) - f(x)| \geq \varepsilon\} \text{ is finite for any } \varepsilon > 0 \text{ and any } x \in X.$$

A sequence $\langle f_n : n \in \omega \rangle$ in $\mathrm{C}_p(X)$ converges to f with respect to $\mathcal{I}$ if

$$\{n : |f_n(x) - f(x)| \geq \varepsilon\} \text{ is in } \mathcal{I} \text{ for any } \varepsilon > 0 \text{ and any } x \in X.$$

If $\langle f_n : n \in \omega \rangle$ in $\mathrm{C}_p(X)$ converges pointwisely to f then it converges with respect to $\mathcal{I}$ as well.

$$f \in \overline{A} \subseteq \mathrm{C}_p(X)$$

Is there a way to describe f via a sequence of elements of A?

Theorem (Cartan 1937)

If $f \in \overline{A} \subseteq \mathrm{C}_p(X)$ then there is a sequence $\{f_n : n \in \omega\} \subseteq A$ and an ideal $\mathcal{I}$ on natural numbers such that $\langle f_n : n \in \omega \rangle$ converges to f with respect to $\mathcal{I}$.

Theorem (Cartan 1937)

If $f \in \overline{A} \subseteq \mathrm{C}_p(X)$ then there is a sequence $\{f_n : n \in \omega\} \subseteq A$ and an ideal $\mathcal{I}$ on natural numbers such that $\langle f_n : n \in \omega \rangle$ converges to f with respect to $\mathcal{I}$.

$\mathrm{C}_p(X)$ has Fréchet-Urysohn property if for any $A \subseteq \mathrm{C}_p(X)$ and function $f \in \overline{A}$ there is a sequence $\langle f_n : n \in \omega \rangle$ in A converging to f.

P. Borodulin-Nadzieja and B. Farkas 2012

$\mathrm{C}_p(X)$ has $\mathcal{I}$-Fréchet-Urysohn property if for any $A \subseteq \mathrm{C}_p(X)$ and function $f \in \overline{A}$ there is a sequence $\langle f_n : n \in \omega \rangle$ in A converging to f with respect to $\mathcal{I}$.

shortly $\mathtt{FU}$, $\mathtt{FU}(\mathcal{I})$

P. Borodulin-Nadzieja and B. Farkas 2012

$\mathrm{C}_p(X)$ has $\mathcal{I}$-Fréchet-Urysohn property if for any $A \subseteq \mathrm{C}_p(X)$ and function $f \in \overline{A}$ there is a sequence $\langle f_n : n \in \omega \rangle$ in A converging to f with respect to $\mathcal{I}$.

Theorem (essentially Borodulin-Nadzieja–Farkas 2012)

$\mathtt{non}(FU(\mathcal{I})) = \mathfrak{p}_{\mathrm{K}}(\mathcal{I})$.

$\texttt{Fin} \subseteq \mathcal{A} \subseteq \mathcal{P}(\omega)$

$\mathcal{A}^*$ has $\texttt{FIP}$ $\Leftrightarrow$ $\mathcal{A}$ has $\texttt{FUP}$ $\Leftrightarrow$ $\mathcal{A} \subseteq \mathcal{I}$ for some $\mathcal{I}$

$\mathcal{A}^*$ has a pseudointersection $\Leftrightarrow$ $\mathcal{A}$ has a pseudounion $\Leftrightarrow$ $\mathcal{A} \leq_K \texttt{Fin}$

Katětov order

$\mathcal{A}_1 \leq_K \mathcal{A}_2$ if there is a function $\varphi\colon \omega \to \omega$ such that $\varphi^{-1}(A) \in \mathcal{A}_2$ for each $A \in \mathcal{A}_1$.

If $\mathcal{A}_1 \subseteq \mathcal{A}_2$ then $\mathcal{A}_1 \leq_K \mathcal{A}_2$.

Pseudointersection numbers $\mathfrak{p}$ and $\mathtt{cov}^*(\mathcal{I})$

$$\mathfrak{p} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{P}(\omega) \text{ has } \mathtt{FUP} \wedge \mathcal{A} \text{ does not have a pseudounion}\}$$

$$\mathtt{cov}^*(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \mathcal{A} \text{ does not have a pseudounion}\}$$

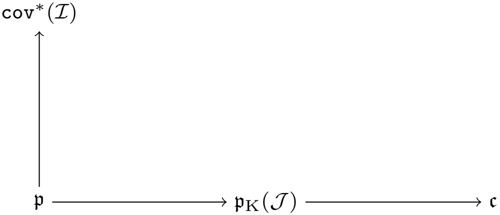
Convention: $\min \emptyset = +\infty$

$\mathtt{Fin}$	$\mathtt{Fin}^2$	$\mathcal{S}$	$\mathcal{ED}$	$\mathtt{Ran}$	$\mathtt{conv}$	$\mathtt{nwd}$
$+\infty$	$\mathfrak{b}$	$\mathtt{non}(\mathcal{N})$	$\mathtt{non}(\mathcal{M})$	$\mathfrak{c}$	$\mathfrak{c}$	$\mathtt{cov}(\mathcal{M})$

Pseudointersection number $\mathfrak{p}_{\mathrm{K}}(\mathcal{J})$

$$\mathfrak{p} = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{P}(\omega) \text{ has } \mathtt{FUP} \wedge \mathcal{A} \not\leq_K \mathtt{Fin}\}$$

$$\mathfrak{p}_{\mathrm{K}}(\mathcal{J}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{P}(\omega) \text{ has } \mathtt{FUP} \wedge \mathcal{A} \not\leq_K \mathcal{J}\}$$



Proposition (J.Š.)

$\min\{\mathfrak{p}_{\mathrm{K}}(\mathcal{I}), \mathtt{cov}^*(\mathcal{I})\} = \mathfrak{p}$.

Problem (Borodulin-Nadzieja–Farkas 2012)

Is $\mathfrak{p}_{\mathrm{K}}(\mathcal{J}) \leq \mathfrak{b}$ for each analytic (P-)ideal $\mathcal{J}$?

Proposition (J.Š.)

If $\mathcal{J}$ is a meager P-ideal then $\mathfrak{p}_{\mathrm{K}}(\mathcal{J}) \leq \mathfrak{b}$.

Theorem (Borodulin-Nadzieja–Farkas 2012)

In the Cohen real model $V^{\mathbb{C}_{\omega_2}}$ the following hold.

(1) *There is a meager ideal $\mathcal{I}$ with $\mathfrak{p}_{\mathrm{K}}(\mathcal{I}) = \omega_2$.*

(2) $\mathfrak{p}_{\mathrm{K}}(\mathcal{J}) = \omega_1$ *for every F_σ ideal $\mathcal{J}$ and every analytic P-ideal $\mathcal{J}$.*

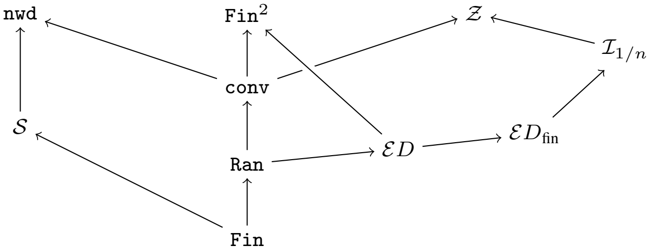
Let $\mathcal{I}_1 \leq_K \mathcal{I}_2$.

If $\mathrm{C}_p(X)$ has $\mathcal{I}_1$-Fréchet-Urysohn property then $\mathrm{C}_p(X)$ has $\mathcal{I}_2$-Fréchet-Urysohn property.

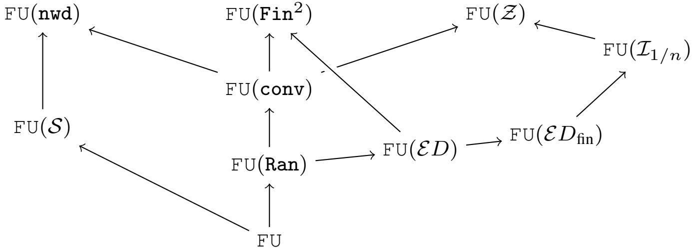
Brendle J. and Flašková J., *Generic existence of ultrafilters on the natural numbers,* Fund. Math. **263** (2017), 201–245.

Hrušák M., *Katětov order on Borel ideals,* Arch. Math. Logic **56** (2017), 831–847.

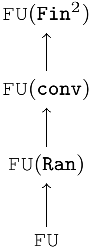
Brendle J., Farkas B. and Verner J., *Towers in filters, cardinal invariants, and Luzin type families,* J. Symbolic Logic **83** (2018), 1013-–1062.



Katětov order



$C_p(X)$



$\mathrm{C}_p(X)$

$$\{V_n:\ n \in \omega\} \subseteq X$$

$\{V_n:\ n \in \omega\}$ is a γ-cover if each $x \in X$ is an element of V_n for all but finitely many n.

$\{V_n:\ n \in \omega\}$ is a γ-cover if $\{n:\ x \notin V_n\}$ is finite for each $x \in X$.

$\{V_n:\ n \in \omega\}$ is an $\mathcal{I}$-γ-cover if $\{n:\ x \notin V_n\}$ is in $\mathcal{I}$ for each $x \in X$.

X is a γ-set if any open ω-cover of X contains a γ-subcover of X.

Topological space X is an $[\Omega, \mathcal{J}\text{-}\Gamma]$-space if for every open ω-cover $\mathcal{V}$ there is a $\mathcal{J}$-γ-cover $\{V_m : m \in \omega\} \subseteq \mathcal{V}$.

Proposition (J.Š.)

(1) *If $\mathrm{C}_p(X)$ has $\mathcal{J}$-Fréchet-Urysohn property then X is an $[\Omega, \mathcal{J}\text{-}\Gamma]$-space.*

(2) $\mathtt{non}(\left[{\Omega \atop \mathcal{J}\text{-}\Gamma}\right]) = \mathfrak{p}_{\mathrm{K}}(\mathcal{J})$.

$$
\begin{array}{c}
\begin{bmatrix}\Omega \\ \texttt{Fin}^2\text{-}\Gamma\end{bmatrix} \\
\big\uparrow \\
\begin{bmatrix}\Omega \\ \texttt{conv}\text{-}\Gamma\end{bmatrix} \\
\big\uparrow \\
\begin{bmatrix}\Omega \\ \texttt{Ran}\text{-}\Gamma\end{bmatrix} \\
\big\uparrow \\
\gamma \\
\\
X
\end{array}
$$

$$\boxed{\underline{\mathrm{Lim}}\{V_n:\ n\in\omega\}=\bigcup_{n\in\omega}\bigcap_{m\geq n}V_m=\{x\in X:\ (\forall^\infty n\in\omega)\ x\in V_n\}}$$

$$\begin{aligned}
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\mathrm{L}^2(\{V_n:\ n\in\omega\}) &= \{\underline{\mathrm{Lim}}\{W_n:\ n\in\omega\}:\ (\forall n\in\omega)\ W_n\in\mathrm{L}^1(\{V_i:\ i\in\omega\})\}\\
\mathrm{L}^\xi(\{V_n:\ n\in\omega\}) &= \{\underline{\mathrm{Lim}}\{W_n:\ n\in\omega\}:\ (\forall n\in\omega)\ W_n\in\bigcup_{\eta<\xi}\mathrm{L}^\eta(\{V_i:\ i\in\omega\})\}
\end{aligned}$$

$\mathrm{L}^{\omega_1}(\{V_n:\ n\in\omega\})$ is the smallest family containing $\mathcal{V}$ and closed under operator $\underline{\mathrm{Lim}}$.

X is a γ-set $\Leftrightarrow$ for any open ω-cover $\mathcal{V}$, $X\in\mathrm{L}^1(\mathcal{V})$

X is a δ-set $\Leftrightarrow$ for any open ω-cover $\mathcal{V}$, $X\in\mathrm{L}^{\omega_1}(\mathcal{V})$

X is a δ-set	$\Leftrightarrow$	for any ω-cover $\mathcal{V}$	$X \in \mathrm{L}^{\omega_1}(\mathcal{V})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
X is an $\left[\Omega, \mathtt{Fin}^{\xi}\text{-}\Gamma\right]$-space	$\Leftrightarrow$	for any ω-cover $\mathcal{V}$	$X \in \mathrm{L}^{\xi}(\mathcal{V})$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
X is an $\left[\Omega, \mathtt{Fin}^{2}\text{-}\Gamma\right]$-space	$\Leftrightarrow$	for any ω-cover $\mathcal{V}$	$X \in \mathrm{L}^{2}(\mathcal{V})$
X is a γ-set	$\Leftrightarrow$	for any ω-cover $\mathcal{V}$	$X \in \mathrm{L}^{1}(\mathcal{V})$

Main result

$$\begin{array}{c}\delta\\ \uparrow\\ \vdots\\ \uparrow\\ \begin{bmatrix}\Omega\\ \mathtt{Fin}^{\xi}\text{-}\Gamma\end{bmatrix}\\ \uparrow\\ \vdots\\ \uparrow\\ \begin{bmatrix}\Omega\\ \mathtt{Fin}^{2}\text{-}\Gamma\end{bmatrix}\\ \uparrow\\ \begin{bmatrix}\Omega\\ \mathtt{conv}\text{-}\Gamma\end{bmatrix}\\ \uparrow\\ \begin{bmatrix}\Omega\\ \mathtt{Ran}\text{-}\Gamma\end{bmatrix}\\ \uparrow\\ \gamma\end{array}$$

X

Theorem 2. *Let X be a Tychonoff-space. If $E = C(X)$, then the following implications are valid.*

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 & \Updownarrow & & \Updownarrow & & \Updownarrow & & \Updownarrow & & \Downarrow & & \Updownarrow\\
X & (\alpha) & \Rightarrow & (\beta) & \Rightarrow & & (\gamma) & & \Rightarrow & (\delta) & \Rightarrow & (\varepsilon)
\end{array}$$

Problem. Is $(\delta) \Rightarrow (\gamma)$ true (in ZFC)? Is there a model of ZFC in which $(\delta) \Rightarrow (\gamma)$ does not hold?

If $\mathfrak{p} = \mathfrak{c}$ then there is an $[\Omega, \mathtt{Ran}\text{-}\Gamma]$-space $A \subseteq \mathcal{P}(\omega)$ which is not a γ-set.

If **CH** holds then there is an $[\Omega, \mathtt{conv}\text{-}\Gamma]$-space $A \subseteq \mathcal{P}(\omega)$ which is not an $[\Omega, \mathtt{Ran}\text{-}\Gamma]$-space.

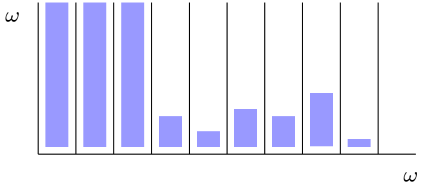
If **CH** holds and $\beta < \alpha$ then there is an $[\Omega, \mathtt{Fin}^{\alpha}\text{-}\Gamma]$-space $A \subseteq \mathcal{P}(\omega)$ which is not an $\left[\Omega, \mathtt{Fin}^{\beta}\text{-}\Gamma\right]$-space.

If **CH** holds then there is a δ-set $A \subseteq \mathcal{P}(\omega)$ which is not an $\left[\Omega, \mathtt{Fin}^{\xi}\text{-}\Gamma\right]$-space.

Katětov ideals

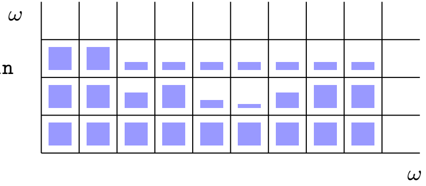
$$A \in \texttt{Fin}^2$$
$$\Updownarrow$$
$$\{n : \{m : (n,m) \in A\} \notin \texttt{Fin}\} \in \texttt{Fin}$$

Fubini product $\texttt{Fin} \times \texttt{Fin}$



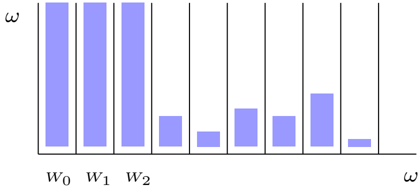
$$A \in \texttt{Fin}^3$$
$$\Updownarrow$$
$$\{n : \{(m,k) : (n,m,k) \in A\} \notin \texttt{Fin}^2\} \in \texttt{Fin}$$

Fubini product $\texttt{Fin} \times (\texttt{Fin} \times \texttt{Fin})$



$$A \in \mathtt{Fin}^2$$
$$\Updownarrow$$
$$\{n : \{m : (n,m) \in A\} \notin \mathtt{Fin}\} \in \mathtt{Fin}$$

Fubini product $\mathtt{Fin} \times \mathtt{Fin}$



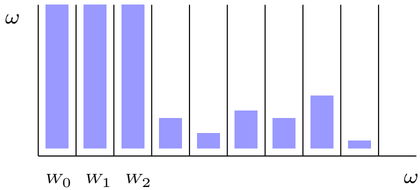
X is an $[\Omega, \mathtt{Fin}^2\text{-}\Gamma]$-space $\Leftrightarrow$ for any ω-cover $\mathcal{V}$ we have $X \in \mathrm{L}^2(\mathcal{V})$.

$\Rightarrow$

- Take ω-cover $\mathcal{V}$.
- Pick $\mathtt{Fin}^2$-γ-cover $\{V_{n,m} : n,m \in \omega\} \subseteq \mathcal{V}$.
- Compute $W_n = \underline{\mathrm{Lim}}\{V_{n,m} : m \in \omega\} = \bigcup_{m\in\omega} \bigcap_{k\geq m} V_{n,k}$.
- Thus $W_n \in \mathrm{L}^1(\mathcal{V})$.
- Note that $X = \underline{\mathrm{Lim}}\{W_n : n \in \omega\}$.

$$A \in \mathtt{Fin}^2$$
$$\Updownarrow$$
$$\{n : \{m : (n,m) \in A\} \notin \mathtt{Fin}\} \in \mathtt{Fin}$$

Fubini product $\mathtt{Fin} \times \mathtt{Fin}$



X is an $[\Omega, \mathtt{Fin}^2\text{-}\Gamma]$-space $\Leftrightarrow$ for any ω-cover $\mathcal{V}$ we have $X \in \mathrm{L}^2(\mathcal{V})$.

$\Leftarrow$

- Take ω-cover $\mathcal{V}$.
- Pick $W_n \in \mathrm{L}^1(\mathcal{V})$ such that $X = \underline{\mathrm{Lim}}\{W_n : n \in \omega\} = \bigcup_{k \in \omega} \bigcap_{k \geq n} W_k$.
- There are $\{V_{n,m} : m \in \omega\} \subseteq \mathcal{V}$ such that $W_n = \underline{\mathrm{Lim}}\{V_{n,m} : m \in \omega\}$.
- Then $\{V_{n,m} : n, m \in \omega\}$ is a $\mathtt{Fin}^2$-γ-cover of X.

Main result

$$
\begin{array}{c}
\delta \\
\uparrow \\
\vdots \\
\uparrow \\
\begin{bmatrix} \Omega \\ \mathtt{Fin}^{\xi}\text{-}\Gamma \end{bmatrix} \\
\uparrow \\
\vdots \\
\uparrow \\
\begin{bmatrix} \Omega \\ \mathtt{Fin}^{2}\text{-}\Gamma \end{bmatrix} \\
\uparrow \\
\begin{bmatrix} \Omega \\ \mathtt{conv}\text{-}\Gamma \end{bmatrix} \\
\uparrow \\
\begin{bmatrix} \Omega \\ \mathtt{Ran}\text{-}\Gamma \end{bmatrix} \\
\uparrow \\
\gamma \\
\\
X
\end{array}
$$

Theorem 2. *Let X be a Tychonoff-space. If $E = C(X)$, then the following implications are valid.*

$$
\begin{array}{lccccccccccc}
E = C(X) & \text{(i)} & \Rightarrow & \text{(ii)} & \Rightarrow & \text{(iii)} & \Leftrightarrow & \text{(iv)} & \Rightarrow & \text{(v)} & \Rightarrow & \text{(vi)} \\
 & \Updownarrow & & \Updownarrow & & \nwarrow\!\!\searrow & & \not\nearrow & & \Downarrow & & \Updownarrow \\
X & (\alpha) & \Rightarrow & (\beta) & \Rightarrow & & (\gamma) & & \Rightarrow & (\delta) & \Rightarrow & (\varepsilon)
\end{array}
$$

Problem. Is $(\delta) \Rightarrow (\gamma)$ true (in ZFC)? Is there a model of ZFC in which $(\delta) \Rightarrow (\gamma)$ does not hold?

If $\mathfrak{p} = \mathfrak{c}$ then there is an $[\Omega, \mathtt{Ran}\text{-}\Gamma]$-space $A \subseteq \mathcal{P}(\omega)$ which is not a γ-set.

If **CH** holds then there is an $[\Omega, \mathtt{conv}\text{-}\Gamma]$-space $A \subseteq \mathcal{P}(\omega)$ which is not an $[\Omega, \mathtt{Ran}\text{-}\Gamma]$-space.

If **CH** holds and $\beta < \alpha$ then there is an $[\Omega, \mathtt{Fin}^{\alpha}\text{-}\Gamma]$-space $A \subseteq \mathcal{P}(\omega)$ which is not an $\left[\Omega, \mathtt{Fin}^{\beta}\text{-}\Gamma\right]$-space.

If **CH** holds then there is a δ-set $A \subseteq \mathcal{P}(\omega)$ which is not an $\left[\Omega, \mathtt{Fin}^{\xi}\text{-}\Gamma\right]$-space.

A brief sketch of the proof of main result

Family $\mathcal{A} \subseteq \mathcal{P}(\omega)$ is $\mathtt{TALL}$ if for any $E \in [\omega]^\omega$ there is $A \in \mathcal{A}$ such that $A \cap E$ is infinite ($\mathcal{A}^*$ does not have a pseudointersection).

$\mathcal{A} \subseteq \mathcal{P}(\omega)$ has $\mathtt{FUP}$, if for any $A_0, \ldots, A_k \in \mathcal{A}$, the set $\omega \setminus \bigcup_{i=0}^k A_i$ is infinite ($\mathcal{A}^*$ has $\mathtt{FIP}$).

Lemma (folklore)

If $A \subseteq \mathcal{P}(\omega)$ is $\mathtt{TALL}$ with $\mathtt{FUP}$, then A is not a γ-set.

Lemma (Galvin–Miller 1984)

If $\mathcal{V}$ is an open ω-cover of $\mathtt{Fin}$ then there is an increasing sequence $\langle k_n : n \in \omega \rangle$ and a family $\{V_n : n \in \omega\} \subseteq \mathcal{V}$ of distinct sets such that $a \in V_n$ for any $a \cap (k_n, k_{n+1}) = \emptyset$.

Theorem

Let $\mathcal{J}$ be $\mathtt{TALL}$. If $\mathfrak{p} = \mathfrak{c}$ then there is an $[\Omega, \mathcal{J}\text{-}\Gamma]$-space $A \subseteq \mathcal{P}(\omega)$ which is not a γ-set.

Proof.

- $\{k_\alpha : \alpha < \mathfrak{c}\} \subseteq {}^\omega\omega$ all increasing functions, $\{c_\alpha : \alpha < \mathfrak{c}\} \subseteq [\omega]^\omega$ all infinite sets.
- We construct $A = \{a_\alpha : \alpha < \mathfrak{c}\} \cup \mathtt{Fin}$ such that
  1. $a_\alpha \subseteq c_\alpha$,
  2. $\{j \in \omega : [k_\alpha(j), k_\alpha(j+1)) \cap a_\alpha \neq \emptyset\} \in \mathcal{J}$.

□

Applications - summary

(1) Gerlits–Nagy problem on γ and δ.

- If $\mathfrak{p} = \mathfrak{c}$ then there is a δ-set which is not a γ-set.

(2) Borodulin-Nadzieja–Farkas pseudointersection number $\mathfrak{p}_{\mathrm{K}}(\mathcal{J})$.

- $\mathtt{non}(\mathrm{FU}(\mathcal{I})) = \mathfrak{p}_{\mathrm{K}}(\mathcal{I})$.
- $\mathtt{non}(\left[{\Omega \atop \mathcal{J}\text{-}\Gamma}\right]) = \mathfrak{p}_{\mathrm{K}}(\mathcal{J})$.
- $\min\{\mathfrak{p}_{\mathrm{K}}(\mathcal{I}), \mathtt{cov}^*(\mathcal{I})\} = \mathfrak{p}$.
-

Fin	$\mathtt{Fin}^2$	$\mathcal{S}$	$\mathcal{ED}$	Ran	conv	nwd
$\mathfrak{p}$	$\mathfrak{p}$	$\mathfrak{p}$	$\mathfrak{p}$	$\mathfrak{p}$	$\mathfrak{p}$	$\mathfrak{p}$

What is next?

Attempting:

- To decide whether $\mathtt{FU}(\mathcal{I})$ are different for different standard critical ideals under **CH**.
- Similarly for $\left[\begin{smallmatrix}\Omega \\ \mathcal{J}\text{-}\Gamma\end{smallmatrix}\right]$.
- To eliminate some assumptions on ideals in counterexamples constructions.
- $\left[\begin{smallmatrix}\Omega \\ \mathcal{Z}\text{-}\Gamma\end{smallmatrix}\right]$ does not imply δ. Does it imply Rothberger property (strong measure zero)?
- Consequences on selection principles $\mathrm{S}_1(\Omega^{\mathtt{ct}}, \mathcal{J}\text{-}\Gamma)$, $\mathrm{S}_1(\mathcal{I}\text{-}\Gamma, \mathcal{J}\text{-}\Gamma)$.
- Is it consistent that $\mathfrak{p}_{\mathrm{K}}(\mathcal{J}) > \mathfrak{p}$ for some nice definable ideal $\mathcal{J}$?

$\vdots$

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Thanks for Your attention!